Numerical solution of 2D unsteady integral boundary layer equations with a discontinuous Galerkin method

H. Özdemir


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Hüseyin Özdemir
Introduction: motivation

- Detailed wind turbine dynamics simulation
- Local aerodynamic forces, structural stresses and deformations

Van Kuik, 2007
Introduction: available tools

• Engineering tools: **not accurate enough**
  - 3D, steady state methods: Blade Element, Momentum (BEM), Vortex line method (AWSM),
  - 2D, steady state methods: XFOIL, RFOIL

• CFD tools (CFX): **too expensive, too much time**
  - Axial-symmetric (1/3\(^{rd}\) of the domain)
  - 2.7 M elements
  - ~2 weeks on 16 node cluster
Introduction: approach

Integral boundary layer method (IBL) + Panel method + Strong quasi-simultaneous viscous – inviscid interaction

Navier-Stokes equations

\[ \frac{\partial p}{\partial y} = 0 \]

Boundary layer equations

\[ \int_{0}^{y_{\infty}} \left[ \text{Cont. eqn.} \times (u^{n+1} - u_{e}^{n+1}) + \text{Mom. eqn.} \times (n + 1) u^{n} \right] \, dy \]

Integral boundary layer equations

Van Dyke, An Album of Fluid Motion, 11th ed., Parabolic Press, 2007, USA
Governing equations

2D, unsteady integral boundary layer eqns:

\[
\frac{\partial F(u)}{\partial t} + \frac{\partial G(u)}{\partial x} = S(u)
\]

\[
F(u) = \begin{bmatrix}
\delta^* \\
\delta^* + \theta \\
\frac{C_T}{U_S}
\end{bmatrix}, \quad G(u) = \begin{bmatrix}
u_e \theta \\
v_e \delta^k \\
v_e C_T
\end{bmatrix},
\]

\[
S(u) = \begin{bmatrix}
\frac{C_f}{2} u_e - (\delta^* + \theta) \frac{\partial u_e}{\partial x} - \delta^* \frac{1}{u_e} \frac{\partial u_e}{\partial t} \\
C_D u_e - 2\delta^k \frac{\partial u_e}{\partial x} - 2\theta \frac{1}{u_e} \frac{\partial u_e}{\partial t} \\
C_T u_e - C_T \frac{\partial u_e}{\partial x} - \frac{2}{U_S} \frac{1}{u_e} \frac{\partial u_e}{\partial t}
\end{bmatrix},
\]

Closure set:

i.e.:

Shape factors

\[
H = \frac{\delta^*}{\theta}, \quad H^* = \frac{\delta^k}{\theta},
\]

Friction coefficient

\[
C_f(H, \theta),
\]

Viscous diffusion coefficient

\[
C_D(H, \theta, H^*(H)),
\]

Slip velocity

\[
U_S(H, H^*(H)),
\]

Shear stress coefficient

\[
C_T(H, H^*(H), U_S).
\]
Governing equations: some analysis

Eigenvalues $\lambda^-$ and $\lambda^+$ for laminar closure relations: system is hyperbolic

Separation point

$\lambda$ becomes negative

Closure relations for the $H$ dependent variables
Numerical method: Discontinuous Galerkin (DG) Method

Solution vector:
\[ \mathbf{u} (\cdot, t) \in U^d, \quad U \equiv L^2(\Omega), \]

Weak formulation
\[ (L(\mathbf{u}(\cdot, t)), \mathbf{v}) = (\mathbf{s}, \mathbf{v}) \]

Approximate solution
\[ \mathbf{u}_h (\cdot, t) \in U_h^d, \quad U_h = \text{span}\{b_{jk}\} \subset U, \quad \forall \mathbf{v} \in U^d, \]
\[ \mathbf{u}(\mathbf{x}, t) = \mathbf{u}_{jk}(t)b_{jk}(\mathbf{x}), \quad \mathbf{u}_{jk}(t) \in L^2(I_t), \quad b_{jk} \in L^2(\Omega), \]

Discrete equation
\[ \int_{\Omega_j} L(\mathbf{u}_h(\mathbf{x}, t)) b_{jm} d\Omega = \int_{\Omega_j} \mathbf{s} b_{jm} d\Omega, \]
\[ \int_{\Omega_j} \frac{\partial \mathbf{u}_{jk}}{\partial t} b_{jk} b_{jm} d\Omega - \int_{\Omega_j} \mathbf{f}_i \frac{\partial b_{jm}}{\partial x_i} d\Omega + \int_{\partial \Omega_j} b_{jm} \mathbf{h}(\mathbf{u}_j, \mathbf{u}_i, \mathbf{n}_j) dS = \int_{\Omega_j} \mathbf{s} b_{jm} d\Omega. \]
**Numerical method: DG Method**

Local Lax-Friedrich flux formula:

\[
h(\bar{u}_j, \bar{u}_l, n_j) = \frac{1}{2} \{ f(\bar{u}_j) + f(\bar{u}_l) - \theta(\bar{u}_l - \bar{u}_j) \}, \quad \theta = \max_{\min(\bar{u}_j, \bar{u}_l) \leq s \leq \max(\bar{u}_j, \bar{u}_l)} |f'(s)|
\]

**Explicit multi-stage Runge-Kutta time integration:** [Cockburn & Shu]

- Set \( U_k^0 = U_k^n; \)
- For \( s = (1, r) \) compute the solution at \( r \) intermediate time stages:
  \[
  U_k^s = \sum_{l=0}^{s-1} \alpha_{sl} W_k^{sl}, \quad W_k^{sl} = U_k^l + \frac{\beta_{sl}}{\alpha_{sl}} \Delta t R_k^l;
  \]
- Set \( U_k^{n+1} = U_k^r. \)

**Slope limiter:** [Cockburn & Shu]

\[
|\bar{u}_h|_{TV(0,1)} \equiv \sum_{1 \leq j \leq N} |\bar{u}_{j+1} - \bar{u}_j|, \quad |\bar{u}_h^{s+1}|_{TV(0,1)} \leq |\bar{u}_h^s|_{TV(0,1)}
\]
Results
model problem: transport equation, continuous initial condition

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \]

\[ u(x, 0) = \sin(2\pi x) \]

periodic b.c.

\[ \Delta x = 1/25 \]
Results

model problem: accuracy

\[ |u_{\Delta t}(x, t) - u_{\Delta t_0}(x, t)| = c\Delta t^\alpha \]
Results

model problem: transport equation, discontinuous initial condition

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \]

\[ u(x, 0) = \begin{cases} 
1, & 0.4 < x < 0.6, \\
0, & \text{otherwise} 
\end{cases} \]

- periodic b.c.
- slope limiter applied
Results

model problem: Burger’s equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = 0$$

$$u(x, 0) = u_0(x) = \frac{1}{4} + \frac{1}{2} \sin(\pi(2x - 1))$$

$$\Delta x = 1/160$$

$t = 0.4$

$t = 1/\pi$
Results: IBL equations

Laminar flow over a flat plate, Re=1e5

\[ u_{edge} = U_\infty \cdot x^m \]
Results: IBL equations

Turbulent flow over a flat plate, Re=1e7
Results: IBL equations

Laminar and turbulent flows over NACA profiles

- Prescribed edge velocity $U_e$ (extracted from XFOIL)
- Dirichlet boundary conditions used
- Initial condition is set
- Only the suction side is considered
- Converging to a steady state problem
- NACA0009 and NACA0012 profiles are used
Results: IBL equations
Laminar flow over NACA0009 profile, Re=1e5
Results: IBL equations
Turbulent flow over NACA0012 profile, Re=1e5

- Introduction
- Governing equations
- Numerical method
- Results
- Conclusions

Graphs showing the comparison between different methods (xfoil, 3rd order DG) for three different parameters (θ, S, H) and their evolution along S.
Results: unsteady simulation

Laminar flow over a flat plate, Re=1e5, with small perturbation in time

$U_e(x,t)$
Results: unsteady simulation
Laminar flow over a cylinder

\[ u_e(x) = V_\infty 2\sin(x) \]
\[ \nu = 1, L = \pi \]
Conclusions and Outlook

- Fully laminar and turbulent flows over a flat plate show good agreement with the literature.
- Flow over NACA profiles are in good agreement up to separation point.
- Non-conservative implementation.
- TVBM slope limiter will be implemented.
- More experimental data needed for 3D unsteady boundary layers and also for rotational effects.
Referenses


